PHYS4150 — PLASMA PHYSICS

LECTURE 4 - PLASMA PROPERTIES: ENERGY AND PLASMA FREQUENCY

Sascha Kempf[*]

G135, University of Colorado, Boulder

Fall 2020

Plasma properties: Energy and Plasma Frequency

---

1 ENERGY

Let us return to the *Debye-Hückel* potential

$$\phi(r) = \frac{Q}{4\pi\epsilon_0} \frac{e^{-r/\lambda_D}}{r}, \tag{1}$$

which we have discussed in the last lecture. $\phi(r)$ describes the *effective* potential of an electron in a plasma, which is not any longer just $\sim r^{-1}$. This results from the necessity for a plasma to maintain charge neutrality by arranging the electrons in a shielding configuration. This means nothing else than that the motion of particles in a plasma is not as random as in a neutral gas and that the energy of a plasma is not just the interior energy of a neutral gas. Following *Debye* and *Hückel* we will now determine the contribution of the correlated plasma particle motion to the energy.

The energy of a system of N electrostatically interacting charged particles is

$$E_e = \frac{N}{2} e\phi_a,$$

where ϕ_a is the potential of the field resulting by the other charged particles acting on the a^{th} particle. We can find ϕ_a by expanding Eq. 1 into a Taylor series and dropping the non-linear terms

$$\phi(r) \approx \frac{e}{4\pi\epsilon_0} \frac{1}{r} - \frac{e}{4\pi\epsilon_0} \frac{1}{\lambda_D}.$$

[*]sascha.kempf@colorado.edu

The first term is the *Coulomb* field of the particle itself, while the second term is the field resulting from the other particles, i.e. ϕ_a, and thus

$$E_e = -\frac{N}{8}\frac{e^2}{\pi\epsilon_0}\frac{1}{\lambda_D} = -\frac{N}{8}\frac{e^2}{\pi\epsilon_0}\left(\frac{n_0 e^2}{\epsilon_0 \mathrm{k_B} T_e}\right)^{1/2},$$
$$= -N\frac{e^3}{8\pi\epsilon_0^{3/2}} n_0^{1/2}\left(\frac{1}{\mathrm{k_B}T_e}\right)^{1/2}$$
$$= -N^{3/2}\frac{e^3}{8\pi\epsilon_0^{3/2}}\left(\frac{1}{V\mathrm{k_B}T_e}\right)^{1/2},$$

where V is the volume. This allows us now to compute the free energy F of the plasma by using

Free energy $F(T,V,N) = U - TS$

$$\frac{E}{T^2} = -\frac{\partial}{\partial T}\frac{F}{T},$$

and

$$F_{plasma} = F_{ideal} - T\int\frac{E_e}{T^2}\,\mathrm{d}T = F_{ideal} - \frac{2}{3}E_e.$$

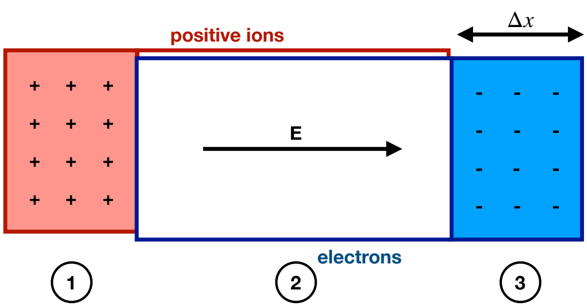
From this follows that the pressure is

$$p(T,V,N) = -\frac{\partial F}{\partial V}|_{T,V} = \frac{N\mathrm{k_B}T}{V} - \frac{E_e}{3V}.$$

2 PLASMA FREQUENCY

Because of charge neutrality a polarization E field will arise from charge imbalances, which eventually will reestablish the neutrality. It is a reasonable assumption that the magnitude of the charge imbalance is proportional to the charge displacement, suggesting that we can describe the response of the plasma to a charge imbalance by a restoring force given by *Hooke's law*, i.e $F = -k\Delta x$.

Let us now consider a slab of electrons of density n_0 and a background of immobile ions of the same density. Now we displace the slab by Δx:



The field strength is given by *Gauss' law*

$$\nabla \mathbf{E}(r) = \frac{n_0 e}{\epsilon_0}$$
$$E = \frac{n_0 e}{\epsilon_0} \Delta x,$$

which gives us the equation of motion for the electrons

$$F = m_e \frac{\mathrm{d}^2}{\mathrm{d}t^2} \Delta x = -eE = -\frac{n_0 e^2}{\epsilon_0} \Delta x,$$
$$0 = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \Delta x + \frac{n_0 e^2}{\epsilon_0 m_e} \Delta x.$$

The resulting equation describes an *harmonic oscillator* $\ddot{x} + \omega_0^2 x = 0$, where

$$\boxed{\omega_0 = \sqrt{\frac{n_0 e^2}{\epsilon_0 m_e}}} \tag{2}$$

is the *plasma frequency*. The resulting plasma oscillations, discovered by *Irving Langmuir* and *Lewi Tonks* in 1929, are called *Langmuir* waves. Note that ω_0 does not depend on the wavelength λ implying that the corresponding phase velocity is proportional to λ and the group velocity vanishes, i.e. there is no charge transport.

Note also that the product of the Debye length and the plasma frequency

$$\lambda_D^2 \omega_0^2 = \frac{\epsilon_0 \mathrm{k_B} T_e}{n_0 e^2} \frac{n_0 e^2}{\epsilon_0 m_e} = \frac{\mathrm{k_B} T_e}{m_e} = c_s^2$$

is the thermal velocity.

3 CRITERIA FOR A PLASMA

We are now prepared to discuss when we will observe plasma effects:

DEBYE SHIELDING: For Debye shielding to happen the number of electrons within the Debye sphere must be large, i.e.

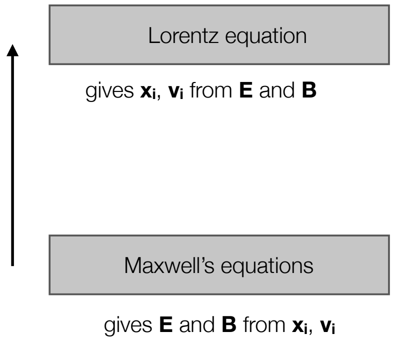
$$N_D = \frac{4}{3} \pi \lambda_D^3 n_e \gg 1.$$

PLASMA WAVES CAN PROPAGATE: For a wave to propagate the number of collisions during the oscillation time scale $1/\omega_0$ must be small.

THE DIMENSION OF THE SYSTEM MUST BE MUCH LARGER THAN λ_D

4 HOW TO SOLVE A PLASMA PROBLEM?

Recall from our first lecture that to solve a plasma problem means to find a good approximation for the system of about 10^{23} coupled equations:



So what can we do here? The general approach is to average over sub-groups of particles and to derive equations of motion for the resulting distributions:

VLASOW: For each specie σ we average over the particles at x with v ($\langle\ \rangle$), which gives us distributions $f_\sigma(\mathbf{x},\mathbf{v},t)$.

TWO FLUIDS APPROACH: For each specie σ we average over the particles at x ($\langle\ \rangle_v$), which gives us distributions for the density $n_\sigma(\mathbf{x},t)$, mean velocity $\mathbf{u}(\mathbf{x},t)$, and the pressure $P_\sigma(\mathbf{x},t)$ (relative to the mean velocity).

MAGNETOHYDRODYNAMICS (MHD): We average over all species at x ($\langle\ \rangle_\sigma$), which gives us the center of mass density $\rho(\mathbf{x},t)$, the center of mass velocity $\mathbf{u}(\mathbf{x},t)$, and the pressure $P(\mathbf{x},t)$ (relative to $\mathbf{u}(\mathbf{x},t)$).